A METHOD OF WORKING OUT THE SUM OF TOTALS OVER BLOCKS CONTAINING A SPECIFIED VARIETY IN THE ANALYSIS OF A $n \times n$ BALANCED LATTICE DESIGN

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A step in the analysis of incomplete block designs is the obtaining of the sum of block totals containing specific varieties. Based upon properties of Latin squares generated by the application of Galois Fields for the general case of $n \times n$ design and cyclic operations upon the columns for the particular case of $p \times p$ design (where p is a prime number) methods are presented which enable such totals to be obtained with relative ease. This is especially useful when the number of characters to be analysed is quite large—the layouts being the same or when the size of lattice employed is big.

Method for $n \times n$ Balanced Lattice Design when n is Power of a Prime

Consider a $n \times n$ balanced lattice design having $n+1$ replications. This may be constructed by the application of Galois Fields as shown by Bose (1939) and Mann (1949). If w is a primitive root of $G.F.$ $(p^m = n)$ the field is $0, 1, w, w^2, w^3, \cdots, w^{n-2}$ of order n. The addition table for these elements gives the $n-1$ orthogonal squares.

An element in the c-th column and r-th row of any such square L_i can be expressed by the relation

$$L_i(c, r) = w^{r-2+i} + w^{c-2};\ c \geqslant 2$$

$$L_i(1, r) = w^{r-2+i}$$

$$i = 0, 1, 2, \cdots, n-2.$$

L_{i+1} is easily obtained from L_i by cyclically permuting the last 1 rows. For a balanced design the number of replications is

$n + 1$. We will designate the different cells with the positions corresponding to columns (c) and rows (r). The cells of the 1st replication are then written as:

$$R_1 = \begin{matrix} 11 & 21 & 31 & \cdots\cdots & n1 \\ 12 & 22 & 32 & \cdots\cdots & n2 \\ 13 & 23 & 33 & \cdots\cdots & n3 \\ .. & .. & .. & & .. \\ .. & .. & .. & & .. \\ 1n & 2n & 3n & \cdots\cdots & nn \end{matrix}$$

The subsequent replications $R_2, R_3, \cdots, R_n$ would then be obtained by superimposing elements of $L_0, L_1, \cdots, L_{n-2}$ upon R_1 while the R_{n+1}-th replication will be obtained by interchanging the rows and columns of R_1. As a matter of convention the elements in the 1st column of the 1st, 2nd, $\cdots$, n-th replication would be kept unchanged—occupying the positions 11, 12, 13, $\cdots$, $1n$ as in R_1. In all cases the rows would be identified with blocks. Let $V(c, r)$ be the cell content of a cell (c, r) and $B(k, r)$ the symbol for the r-th block in k-th replication.

Arrange the blocks (block symbols or block totals) to form an $n \times n$ square in such a way that the element $B(k, r)$ is in cell no. (k, r) of this square. It may be shown that there exists a set of $n - 1$ orthogonal Latin squares L'_{c-2} which when superimposed upon the $n \times n$ square gives the blocks $B(k, r)$ all corresponding to symbol, say α, of the Galois field where $B(k, r)\,s$ contain a particular $V(c, r)$—the initial value of the Bs being $B(1, r)$ having the symbol α. However, in practice the L' squares are not necessary. Write the B-values for the last replication R_{n+1} by entering block 1 in column 1, block 2 in column 2 and so on. Enter the 1st column of L_0 in all the columns of the $n \times n$ square of the Bs with 0 in the 1st row, 1 in the second row, w in the 3rd row and so on. Superimpose L_0 upon another $n \times n$ square with cells numbered (c, r). To find the Bs corresponding to a $V(c, r)$ note the Galois Field symbol in cells in the order $(1, r)$, (c, r) $(c - 1, r)$, $(c - 2, r)$, $\cdots$, $(2, r)$, (n, r), $(n - 1, r)$, $\cdots$, $(c + 1, r)$. Then these Galois Field symbols in the 1st, 2nd, 3rd, $\cdots$, n-th columns respectively of the $n \times n$ square for the Bs gives the required Bs. To these Bs is added the B from the additional row at the bottom and in c-th column. The following example for a 4×4 design would illustrate the method:

$L_0 =$

11 (0)	21 (1)	31 (2)	41 (3)
12 (1)	22 (0)	32 (3)	42 (2)
13 (2)	23 (3)	33 (0)	43 (1)
14 (3)	24 (2)	34 (1)	44 (0)

FIG. 1

L_0, the basic Latin square obtained from the addition table for $GF(2^2) = 0, 1, x, x+1$ and putting $x = 2$, is given in Fig. 1.

B-values and G.F. numbers of 1st column of L_0

(0) 11	(0) 21	(0) 31	(0) 41
(1) 12	(1) 22	(1) 32	(1) 42
(2) 13	(2) 23	(2) 33	(2) 43
(3) 14	(3) 24	(3) 34	(3) 44
51	52	53	54

FIG. 2

Figure 2 gives the table for *B*-values. The *G.F.* numbers 0, 1, 2, 3 are entered along the 1st, 2nd, 3rd, 4th rows in each cell respectively.

Let us find out the *B*s for, say, $V(3, 4)$. In the L_0 table Fig. 1 in the 4th row we get the cell nos. (14), (34), (24), (44) with *G.F.* numbers 3, 1, 2, 0 respectively. Looking upon these *G.F.* numbers in the 1st, 2nd, 3rd, 4th columns of Fig. 2 we get the *B*s as $B(14)$, $B(22)$, $B(33)$, $B(41)$. To these is added the B value $B(53)$ which is directly below the element (34) in the extra row. Thus the *B*s corresponding to $V(34)$ are $B(14)$, $B(22)$, $B(33)$, $B(41)$, $B(53)$. If the *B*s represent the block totals then the total of these Bs is the required total for these blocks which contain a variety $V(c, r)$.

A Modified Procedure when n is a Prime Number p

An alternate method but simpler than the one described above may be adopted when n is a prime number equal to p. The different replications $R_1, R_2, \cdots, R_p$ are to be formed from the rule that the rows of R_{s+1} are the diagonals of R_s the elements in the 1st column remaining unchanged. We adopt the convention that the diagonals are to be taken downwards. The last replication R_{p+1} is obtained as before. An example for 3×3 design is given in Fig. 3. The downward diagonal of R_1 with starting element (11) is (11), (22), (33).

R_1			$B(1, r)$	R_2			$B(2, r)$	R_3			$B(3, r)$
11	21	31	11	11	22	33	21	11	23	32	31
12	22	32	12	12	23	31	22	12	21	33	32
13	23	33	13	13	21	32	23	13	22	31	33

R_4			$B(4, r)$
11	12	13	41
21	22	23	42
31	32	33	43

B-values		
11	21	31
12	22	32
13	23	33
41	42	43

Fig. 3

d-values		
3	2	1
11	21	31
12	22	32
13	23	33

Fig. 4

In R_2 these elements form the rows with (11) as leading element. Likewise, the diagonal with leading element (12) is (12), (21), (33). In R_3 these elements form the row with (12) as leading element. Next, arrange the $B(c, r)$ table as in the general case. To simplify the procedure of recording the totals arrange another $p \times p$ square and enter 'd' numbers $p, p-1, p-2, \cdots, 3, 2, 1$ above the square against columns 1, 2, 3 $\cdots, p-2, p-1, p$ respectively as shown in Fig. 4 for a 3×3 design. In the various cells enter the numbers (c, r) for recording the totals corresponding to $V(c, r)$. The 'd' numbers indicate the order in which the various cells for the B-values have to be taken for adding. To record a total in a cell (c, r) with respect to a $V(c, r)$ note the 'd' number. The cells to be added are then $B(1, r)$, $B(2, r+d)$, $B(3, r+2d)$, $B(4, r+3d)$, .., $B(s, r+\overline{s-1}.d)$ ...., *i.e.*, the difference between two consecutive *Bs* being d rows. Due care is to be taken to reduce the series $r+\overline{s-1}.d$ to $(\text{mod}\, p)$. Finally, add the B-value from the last row which is in column c. For a 3×3 design Fig. 4 gives the B-values and the additional 3×3 square in which the totals are to be recorded. To get B-values corresponding to, say, $V(2, 3)$ we look up the d-value above cell (23). This is 2. We, therefore, start with B (13) and record the diagonal elements with difference 2 rows between consecutive B-values thus getting B (22) and B (31). To these we add, as in the general case, B (42) from the extra row for the last replication. The values B (13), B (22), B (31), B (42) agree with what we actually obtain from Fig. 3.

It may be pointed out that the general method is applicable also to cases where $n = p_1^{a_1} . p_2^{a_2} \cdots, p_s^{a_s}$. The manner of obtaining the Latin squares in such cases have been shown by Bose (1939) and Mann (1949). The procedure thereafter is identical to one suggested above. Cases where the number of replications are less than $n+1$ are treated easily by merely leaving out the cells for these replications in $B(k, r)$ tables. Needless to say that this covers the case of simple lattice as well.

SUMMARY

Methods to obtain the sum of the totals over those blocks which contain a particular variety with relative ease in a balanced lattice design with $n \times n$ varieties where n is any number have been described. The procedure involves the arrangement of the block totals of the first n replications in the form of a $n \times n$ square and the last replication block total in an additional row at the bottom of the $n \times n$ square and the

summation over these blocks which are indicated by the Galois Field numbers of the basic $n \times n$ Latin square. A simpler procedure is provided for the special case where n is a prime number.

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